R is a complete ordered field including two special elements 0, 1 (zero, one) with +, x (addition, multiplication), and order

$x \leq y$ ($\Leftrightarrow x < y$ or $x = y$, strictly smaller or equal)

i.e. $x < y \Leftrightarrow x \leq y$ and $x \neq y$.

SATISFYING the following "axioms" I, II, and III:

I (field axiom) :

(1) $(x+y)+z = x+(y+z)$ for all x, y, z in $\mathbb{R}$ (i.e. $\forall x, y, z \in \mathbb{R}$);

(ii) $x + 0 = 0 + x = x$, $\forall x \in \mathbb{R}$;

(iii) $\forall x \in \mathbb{R}$, $\exists (-x) \in \mathbb{R}$ s.t. $x + (-x) = (-x) + x = 0$;

(iv) $x + y = y + x$, $\forall x, y \in \mathbb{R}$;

(v) $0 \neq 1$;

(vi) $(xy)z = x(yz)$ $\forall x, y, z \in \mathbb{R}$ (here and throughout xy means $x \times y$).

(vii) $x1 = 1x = x$ $\forall x \in \mathbb{R}$;

(viii) $\forall x \in \mathbb{R}\setminus\{0\}$, $\exists x^{-1} \in \mathbb{R}\setminus\{0\}$ s.t. $x x^{-1} = x^{-1} x = 1$;

(ix) $xy = yx$ $\forall x, y \in \mathbb{R}$;

(x) $x(y+z) = xy + xz$ & $(y+z)x = yx + zx$ $\forall x, y, z \in \mathbb{R}$.

II. Order-field axiom

II.1) trichotomy property: $\forall x, y \in \mathbb{R}$, exactly one and only one of the following cases will happen: ① $x < y$,

② $x = y$

③ $y < x$ (i.e. $x > y$)

(so $x \leq y$, $y \leq x \Rightarrow x = y$; why? — Exercise!)

(N-I) $X = \{0\}$ satisfies I except (v).

Apart from $\mathbb{R}$, there are many other systems satisfying Axiom I, e.g.,

$\mathbb{Q}$ (of rational numbers), satisfies I, II.

(N- I & II). With $+$ & $\times$ defined below for $\{0,1\}$ $(0 \neq 1)$ it is not possible to define an order $<$ satisfying all I & II

$0+1 = 1+0 = 1,$

$1+1 = 0.$ $(0\cdot 0 = 0, \ 0\cdot 1 = 1\cdot 0 = 0, \ 1\cdot 1 = 1)$

equivalently $\{0,1\}$ to be defined as $\mathbb{Z}/\text{mod. } 2$:

$x \sim y$ iff $x-y$ divisible by 2

$(x, y \in \mathbb{Z}$, integers$)$

Then $\{0,1\}$ $(= \mathbb{Z}/\text{mod. } 2)$ does Not satisfy II (Should $1<0$ then $0 = 1+1 < 0+1 = 1$ leads a contradiction; therefore $1>0$ (as $1 \neq 0$, & make use of the trichotomy property) However $1>0$ also leads a contradiction in the system $\{0,1\}$ because of the translation property.

Note

1. Uniqueness in (ii), (iii), (vii), (viii), e.g. for (viii):

Suppose $x \in \mathbb{R}\setminus\{0\}$, $x^{-1} \in \mathbb{R}$ and $x' \in \mathbb{R}$ s.t.

$x x^{-1} = x^{-1} x = 1$ &

$x x' = x' x = 1$.

Then

$x^{-1} = 1 \cdot x^{-1} = (x' x) x^{-1} = x' \cdot 1 = x'$

II. 2) transitive : $x<y,\ y<z \Rightarrow x<z$

II. 3) compatible w.r.t. $+$

& w.r.t. positive multiplication, i.e.

$x<y \Rightarrow x+z<y+z\ \forall z\in\mathbb{R}$ ("translation invariant")

$x<y$ & $0<\lambda \Rightarrow \lambda x<\lambda y$

III Order-Complete axiom:

If A is a nonempty subset of $\mathbb{R}$ and $\lambda\in\mathbb{R}$ such that $a\le\lambda\ \forall a\in A$ (i.e. λ is an upper bound of A) then there exists $\lambda_0\in\mathbb{R}$, the smallest upper bound of A, i.e.

1) λ_0 is an upper bound of A: $a\le\lambda_0\ \forall a\in A$,

2) $\lambda_0\le\lambda$ whenever λ is an upper bound of A.

Note. 2) can be stated equivalently as

2^*) If $\lambda<\lambda_0$ then $\exists\, a\in A$ s.t. $a>\lambda$.

(i) – (iv) can be summarized as

$(\mathbb{R}, +)$ is a commutative (Abelian) group.

(vi) – (ix) equivalent to saying that

$(\mathbb{R}\setminus\{0\}, \cdot)$ is a commutative group

(i) – (x) can be stated as $\mathbb{R}$ is a field

I & II together means that $\mathbb{R}$ is an ordered field.

Remark. The property $1 \cdot 0 = 0$ can be proved by the other properties of $\mathbb{R}$:

$$1 \cdot 0 = 1 \cdot (0+0) = 1 \cdot 0 + 1 \cdot 0$$

and so $1 \cdot 0 = 0$ (by adding the "inverse" (w.r.t $+$) $-(1 \cdot 0)$ of $(1 \cdot 0)$)

Notes & Ex. (not yet need III).

1. Uniqueness

2. Usual "Cancellation Laws" hold (in $\mathbb{R}$:) only need I

$$x+z=y+z \Rightarrow x=y$$

$$xz=yz,\ z\neq 0 \Rightarrow x=y$$

3. $(-1)x=-x$ ($\because$ LHS has the property $(-1)x+x=((-1)+1)x=0.x=0$ — now apply the uniqueness of "additive inverse")

4. $(-1)(-1)=1$ ($\because$ LHS has the property $(-1)(-1)+(-1)=((-1)+1)\cdot(-1)=0\cdot(-1)=0$ so $(-1)(-1)$ is the additive inverse of -1)

5. $x<y \Leftrightarrow -y<-x$ $\left(\Leftrightarrow 0<y-x \overset{def}{=} y+(-x)\right)$

$(x<y \Rightarrow x+(-x)<y+(-x) \Rightarrow 0<y+(-x)$.

6. $0<x$ & $0<y \Rightarrow 0<x<x+y$.

7. $x_i<y_i\ (i=1,2) \Rightarrow x_1+x_2<y_1+y_2$

8. If $x\neq 0$ then $0<x^2\ (=(-x)^2)$, where $x^2 \overset{def}{=} x.x$. (Proof. Consider two cases; $0<x$ & $0<(-x)$ (noting $x\neq 0$, the assumption).

Natural Numbers & Math. Induction (MI).
(Only use axioms I & II).

Definition. $\mathbb{N}$ is defined to be the smallest subset of $\mathbb{R}$ s.t.

(i) $1 \in \mathbb{N}$

(ii) $\mathbb{N}$ is inductive :

$$x \in \mathbb{N} \Rightarrow x+1 \in \mathbb{N}.$$

Math Induction (MI). Suppose $P(n)$ is a statement for any $n \in \mathbb{N}$ such that

$P(1)$ is true;

$P(n+1)$ is true whenever $P(n)$ is true.

Then $P(n)$ is true $\forall n \in \mathbb{N}$.

Proof. Let $Z \overset{def}{=} \{n \in \mathbb{N} : P(n) \text{ is true}\}$.
Then Z is inductive and $1 \in Z$.
Since $Z \subseteq \mathbb{N}$ it follows from the "smallest property" stated in the def of $\mathbb{N}$ that $Z = \mathbb{N}$ and so $P(n)$ is true $\forall n \in \mathbb{N}$

Extended MI. Suppose that

(i*) $P(1)$ is true

(ii*) If $n \in \mathbb{N}$ such that

$P(k)$ is true for all $k = 1, \cdots, n$

then $p(n+1)$.

Then $p(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $Q(n)$ denote the combined statement of $P(1), P(2), \cdots, P(n)$. Thus $Q(1)$ is the same as $P(1)$, and note that $Q(n)$ holds means that $P(1), P(2), \cdots, P(n)$ hold. The given (i*) and (ii*) can be restated as

(i) $Q(1)$ is true

(ii) $Q(n)$ is true $\Rightarrow Q(n+1)$ is true.

Now apply MI to $Q(n)$.

Cor 1. Let $X \subseteq \mathbb{N}$ be a finite set (say $\#(X) = n$, i.e. there are $n (\in \mathbb{N})$ many distinct elements in X). Then

X has a (the) greatest element

" " " smallest element

Proof. By M.I. (Exercise)

Cor 2. 1 is the smallest element in $\mathbb{N}$ and $\mathbb{N}$ is an infinite set (that is, not a finite set).

Proof. Let

$$\mathbb{N}_0 = 1 \cup \{n \in \mathbb{N} : 1 < n\}$$

Then, as $\mathbb{N}_0$ is seen to be inductive and contains 1, one has $\mathbb{N}_0 = \mathbb{N}$ and so any $n \in \mathbb{N} \setminus \{1\}$ is bigger than 1.

For the 2nd assertion, note that any $n \in \mathbb{N}$ is smaller than $n+1$ (which is also in $\mathbb{N}$), so $\mathbb{N}$ does not have a largest element, and hence $\mathbb{N}$ must not be finite by Cor 1.

Cor 3. Let $2 := 1+1$, $3 := 2+1$ etc. Then

$$1 < 2 < 3 < 4 < \cdots$$

(so all distinct) and, $\forall n \in \mathbb{N}$,

$$(n, n+1) \cap \mathbb{N} = \emptyset,$$ thus informally, $\mathbb{N} = \{1, 2, 3, \cdots\}$.

proof. Let $\mathbb{N}_1 := \{1\} \cup \{n \in \mathbb{N} : 2 \leq n\}$.

Similar as before $\mathbb{N} = \mathbb{N}_1$ and so $\nexists\, n \in \mathbb{N}$ s.t. $1 < n < 2$ (since any $n \in \mathbb{N}$ should be either 1 or $2 \leq n$). Similarly $\mathbb{N}_2 = \mathbb{N}$ where

$$\mathbb{N}_2 := \{1, 2\} \cup \{n \in \mathbb{N} : 3 \leq n\}$$

and so $(2, 3) \cap \mathbb{N} = \emptyset$. In general, $\forall N \in \mathbb{N}$ one has $(N, N+1) \cap \mathbb{N} = \emptyset$ (you are invited to check this via extended MI).

Well-Order Principle for N. Let X be a nonempty set of natural numbers.
(I) If X is finite then it has the smallest and the largest elements.
(II) X has the largest elements if and only if (iff) there exists a natural number n dominating (bigger than or equal to) every members of X. [Hint on Proof: induction over n].

Let Z denote the set of all integers, that is Z: = { n: n =0, or n is a natural number or -n is a natural number.}.

Generalised Well-Order Principle for Z. Let X be a nonempty subset of Z.

(I) Let n be a natural number such that $-n < x$ for all x in X (such n does exist in the case when Z is finite).
Then {n+x: x in X} is a subset of N.
(II) If X is finite then it has the smallest and the largest elements.
(III) X is finite iff there exist natural numbers n and m such that $-n < x < m$ for all x in X.